Let $x_1, x_2, \ldots, x_n$ be a random sample of n observations from a population having pdf $f(x, \theta)$, $\theta \in \Theta$. It is assumed that n is large. The likelihood function of the sample observations is

$$L = f(x_1, x_2, \ldots, x_n, \theta)$$

$$= f(x_1, \theta) \cdot f(x_2, \theta) \cdots f(x_n, \theta)$$

$$= \prod f(x_i, \theta)$$

The maximum likelihood estimator of θ is obtained by solving the equation

$$\frac{\partial \log L}{\partial \theta} = 0 \quad \text{Which makes}$$

$$\frac{\partial^2 \log L}{\partial \theta^2} < 0 .$$

Again,

it is also known the maximum likelihood estimator is asymptotically normally distributed and $E\left(\frac{\partial \log L}{\partial \theta}\right) = 0$ and

$$V\left(\frac{\partial \log L}{\partial \theta}\right) = E\left(\frac{\partial \log L}{\partial \theta}\right)^2$$

$$= -E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right)$$

$\therefore$ If the sample size is large, then

$$\left(\frac{\partial \log L}{\partial \theta}\right) \Big/ \sqrt{V\left(\frac{\partial \log L}{\partial \theta}\right)} \sim N(0,1).$$

$\therefore$ The $100 \times (1-\alpha)\%$ confidence interval for θ is obtained from the following equation:

$$P_\theta\left\{ \left| \frac{\frac{\partial \log L}{\partial \theta}}{\sqrt{V\left(\frac{\partial \log L}{\partial \theta}\right)}} \right| \leq Z_{\alpha/2} \right\} = 1-\alpha$$

Here, $Q = \frac{\partial \log L}{\partial \theta}$ and

$$P\{q_2 < Q < q_1\} = 1-\alpha .$$

The values of q_1 and q_2 are obtained from normal probability table.

Let $x_1, x_2, \cdots, x_n$ be a random sample of n observations from a population with pdf $f(x,\theta) = \frac{1}{\theta} e^{-x/\theta}$. Find $100\times(1-\alpha)\%$ confidence interval, assuming n is large.

$$L = \prod f(x_i, \theta)$$

$$= \prod \frac{1}{\theta} e^{-x_i/\theta}$$

$$= \frac{1}{\theta^n} e^{-\sum x_i/\theta}$$

$$\Rightarrow \log L = n \log\left(\frac{1}{\theta}\right) - \frac{\sum x_i}{\theta}$$

$$\Rightarrow \frac{\partial \log L}{\partial \theta} = n\theta \cdot \left(-\frac{1}{\theta^2}\right) + \frac{\sum x_i}{\theta^2}$$

$$= -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2} = \frac{n}{\theta}\left(\frac{\bar{x}}{\theta} - 1\right)$$

$$\Rightarrow \frac{\partial^2 \log L}{\partial \theta^2} = \frac{n}{\theta^2} - 2\frac{\sum x_i}{\theta^3}$$

$$= -\frac{n}{\theta^2}\left(\frac{2\bar{x}}{\theta} - 1\right)$$

$$\therefore -E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = +\frac{n}{\theta^2}\left(\frac{2}{\theta}E(\bar{x}) - 1\right)$$

Now, $E(\bar{x}) = \frac{1}{n}\sum E(x_i) = \frac{1}{n} \cdot n\theta = \theta$

$$\therefore -E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = \frac{n}{\theta^2}$$

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$$\frac{\frac{\partial \log L}{\partial \theta}}{\sqrt{V\left(\frac{\partial \log L}{\partial \theta}\right)}} = \frac{\frac{n}{\theta^2}\{\bar{x}-\theta\}}{\sqrt{n/\theta^2}} = \frac{\sqrt{n}}{\theta}\{\bar{x}-\theta\}$$

$$= \frac{\bar{x}-\theta}{\theta/\sqrt{n}}$$

$\therefore$ $100 \times (1-\alpha)\%$ confidence interval for θ of large n is given

$$P_\theta\left\{-Z_{\alpha/2} \leq \frac{\bar{x}-\theta}{\theta/\sqrt{n}} \leq Z_{\alpha/2}\right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{-Z_{\alpha/2} \leq \sqrt{n}\left\{\frac{\bar{x}}{\theta}-1\right\} \leq Z_{\alpha/2}\right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{-Z_{\alpha/2}/\sqrt{n} \leq \frac{\bar{x}}{\theta}-1 \leq Z_{\alpha/2}/\sqrt{n}\right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{1-Z_{\alpha/2}/\sqrt{n} \leq \frac{\bar{x}}{\theta} \leq 1+Z_{\alpha/2}/\sqrt{n}\right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{\frac{(1-Z_{\alpha/2}/\sqrt{n})}{\bar{x}} \leq \frac{1}{\theta} \leq \frac{(1+Z_{\alpha/2}/\sqrt{n})}{\bar{x}}\right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{\frac{\bar{x}}{(1-Z_{\alpha/2}/\sqrt{n})} \leq \theta \leq \frac{\bar{x}}{(1+Z_{\alpha/2}/\sqrt{n})}\right\} = 1-\alpha$$

Hence, $100 \times (1-\alpha)\%$ confidence interval of θ for large n is $\left(\frac{\bar{x}}{1-Z_{\alpha/2}/\sqrt{n}}, \frac{\bar{x}}{1+Z_{\alpha/2}/\sqrt{n}}\right)$

Obtain ~~95%~~ $100 \times (1-\alpha)\%$ confidence limits for the parameter p in $B(n,p)$ using large sample.

$$L = \prod f(x_i, p)$$

$$= \prod \binom{n}{x_i} p^{x_i} (1-p)^{-x_i+n}$$

$$= \left\{\prod \binom{n}{x_i}\right\} p^{\sum x_i} (1-p)^{-\sum x_i + n^2}$$

$$\cancel{\Rightarrow \frac{\partial \log L}{} = \log\left\{\prod \binom{n}{x_i}\right\} + \sum x_i \log p + \sum x_i \log (1-p)}$$

$$\cancel{= c + \sum x_i \log (}$$

$$\Rightarrow \log L = \log\left\{\prod \binom{n}{x_i}\right\} + \sum x_i \log p + \left(-\sum x_i + n^2\right) \log (1-p)$$

$$\Rightarrow \frac{\partial \log L}{\partial p} = + \frac{\sum x_i}{p} - \frac{-\sum x_i + n^2}{1-p}$$

$$\cancel{\Rightarrow \partial \log} = \frac{n\bar{x}}{p} - \frac{-n\bar{x} + n^2}{1-p}$$

$$= n\left(\frac{\bar{x}}{p} - \frac{n - \bar{x}}{1-p}\right)$$

$$\Rightarrow \frac{\partial^2 \log L}{\partial p^2} = n\left(-\frac{\bar{x}}{p^2} - \frac{(n-\bar{x})}{(1-p)^2}\right)$$

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$$\Rightarrow -E\left(\frac{\partial^2 \log L}{\partial p^2}\right) = -n\left(-\frac{E(\bar{x})}{p^2} \mp \frac{n - E(\bar{x})}{(1-p)^2}\right)$$

Now, $E(\bar{x}) = \frac{1}{n} \sum E(x_i)$

$$= \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot np$$

$$\therefore -E\left(\frac{\partial^2 \log L}{\partial p^2}\right) = -n\left(-\frac{np}{p^2} \mp \frac{n-np}{(1-p)^2}\right)$$

$$= -n\left(-\frac{n}{p} \mp \frac{n}{1-p}\right)$$

$$= n^2\left(\frac{1}{p} + \frac{1}{1-p}\right)$$

$$= n^2 \frac{1}{p(1-p)}$$

$$\therefore \frac{\partial \log L}{\partial p} \Big/ \sqrt{V\left(\frac{\partial \log L}{\partial p}\right)} = \frac{\frac{\bar{x}}{p} - \frac{n-\bar{x}}{-p+1}}{\frac{n}{\sqrt{p(1-p)}}}$$

$$= \frac{\bar{x} - \cancel{p\bar{x}} - np + \cancel{p\bar{x}}}{n\sqrt{p(1-p)}}$$

$$= \frac{\cancel{n}(\bar{x} - np)}{\cancel{n}\sqrt{p(1-p)}}$$

$$= \frac{\bar{x} - np}{\sqrt{p(1-p)}}$$

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$$P_\theta\left\{-Z_{\alpha/2} \le \frac{\bar{x}-np}{\sqrt{p(1-p)}} \le Z_{\alpha/2}\right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{ (\bar{x}-np)^2 \le Z_{\alpha/2}^2 \{\sqrt{p(1-p)}\}^2 \right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{ \bar{x}^2 - 2\bar{x}np + n^2p^2 \le Z_{\alpha/2}^2 \{p\cancel{(1-2p+p^2)}\} \right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{ \bar{x}^2 + (-2\bar{x}n - Z_{\alpha/2}^2)p + (n^2+Z_{\alpha/2}^2)p^2 \le 0 \right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{ (n^2+Z_{\alpha/2}^2)p^2 - (2\bar{x}n + Z_{\alpha/2}^2)p + \bar{x}^2 \le 0 \right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{ p = \frac{(2\bar{x}n + Z_{\alpha/2}^2) \pm \sqrt{\cancel{(n^2+Z_{\alpha/2}^2)^2} - 4 (n^2+Z_{\alpha/2}^2)\bar{x}^2}}{2(n^2+Z_{\alpha/2}^2)} \right\} = 1-\alpha$$

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