Rao Blackwell Theorem

Let $x_1, x_2, \ldots x_n$ be a random sample of size n drawn from the population having the p.d.f $f(x, \theta)$. Let $Y_1 = u_1(x_1, \ldots$ be a sufficient statistic for θ and $Y_2 = u_2(x_1, x_2, \ldots x_n)$ be an unbiased estimate of θ. Then $E(Y_2/Y_1) = \phi(y_1)$ (say) is a function of the sufficient statistic Y_1 for θ and it is unbiased for θ and it has variance less than that of Y_2.

Proof:-

Let $g(y_1, y_2, \theta)$ be the joint p.d.f of the sufficient statistic Y_1 and unbiased estimator Y_2 and $g_1(y_1, \theta)$, $g_2(y_2, \theta)$ be the marginal p.d.f of Y_1 and Y_2 respectively and $h(y_2/y_1)$ be the conditional p.d.f. of Y_2 for given y_1.

Hence we have

$$E(Y_2/Y_1) = \int_{-\infty}^{\infty} y_2\, h(y_2/y_1)\, dy_2$$

$$= \int_{-\infty}^{\infty} y_2 \frac{g(y_1, y_2, \theta)}{g_1(y_1, \theta)}\, dy_2$$

$$= \frac{1}{g_1(y_1, \theta)} \int_{-\infty}^{\infty} y_2\, g(y_1, y_2, \theta)\, dy_2$$

$$= \phi(y_1).$$

Where $\phi(y_1)$ does not depend upon θ because $h(y_2/y_1)$ does not depend on θ.

Now

$$E\{\phi(y_1)\} = \int_{-\infty}^{\infty} \phi(y_1)\, g_1(y_1, \theta)\, dy_1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_2\, g(y_1, y_2, \theta)\, dy_1\, dy_2$$

$$= \int_{-\infty}^{\infty} y_2 \left\{\int_{-\infty}^{\infty} g(y_1, y_2, \theta)\, dy_1\right\} dy_2$$

$$= \int_{-\infty}^{\infty} y_2\, g_2(y_2, \theta)\, dy_2 = \theta$$

$Var(Y_2) = E\{Y_2 - \theta\}^2$

$= E\left[\{Y_2 - \varphi(Y_1)\} + \{\varphi(Y_1) - \theta\}\right]^2$

$= E\{Y_2 - \varphi(Y_1)\}^2 + E\{\varphi(Y_1) - \theta\}^2 + 2E\left[\{Y_2 - \varphi(Y_1)\}\{\varphi(Y_1) -$

Now, $E\left[\{Y_2 - \varphi(Y_1)\}\{\varphi(Y_1) - \theta\}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{Y_2 - \varphi(Y_1)\}\{\varphi(Y_1) - \theta\}\, g(Y_1, Y_2, \theta)\, dY_1\, d$

$= \int_{-\infty}^{\infty}\{\varphi(Y_1) - \theta\}\, g_1(Y_1, \theta)\left[\int_{-\infty}^{\infty}\{Y_2 - \varphi(Y_1)\}\, h(Y_2/Y_1)\, dY_2\right]$

But $\varphi(Y_1)$ is the mean of the conditional p.d.f $h(Y_2/Y_1)$ hen

$\int\{Y_2 - \varphi(Y_1)\}\, h(Y_2/Y_1)\, dY_2 = 0$

Hence $Var(Y_2) = E\{Y_2 - \varphi(Y_1)\}^2 + E\{\varphi(Y_1) - \theta\}^2$

$= E\{Y_2 - \varphi(Y_1)\}^2 + Var\{\varphi(Y_1)\}$

$\therefore Var(Y_2) \geq Var\{\varphi(Y_1)\}$

Hence the theorem.