Necessary and sufficient condition for sufficiency

The necessary and sufficient condition for an estimator to be sufficient for a parameter is provided by the factorization theorem due to Neyman is as follows :-

The estimator $T = t(x_1 x_2 \dots x_n)$ is sufficient for θ if and only if the joint p.d.f. L (say) of the sample values can be expressed is the form

$$L = g_\theta[t(x)] \cdot h(x).$$

where $g_\theta[t(x)]$ depends on θ and x only through the value of $t(x)$ and $h(x)$ is independent of θ.

Proof :

Necessari condition

Let the statistic T_1 be sufficient for the parameter θ. If we consider any additional $(n-1)$ statistics $T_2, T_3, \dots T_n$ which are algebraically unrelated to T_1, then the conditional distribution of $T_2, T_3, \dots T_n$ for given $T_1 = t_1$ must be independent of θ. Hence the joint p.d.f of $T_1, T_2, \dots T_n$ will be of the form.

$$\varphi_\theta(t_1 t_2 \dots t_n) = g_{1\theta}(t_1)\ h_1(t_2 t_3 \dots t_n / t_1)$$

where $g_{1\theta}(t_1)$ is the marginal p.d.f. of T_1 and $h(t_2 t_3 \dots t_n / t_1)$ is the conditional p.d.f of $T_2 T_3 \dots T_n$ given $T_1 = t_1$.

Now let us consider the one to one transformation.

$$(t_1 t_2 \dots t_n) \rightarrow (x_1 x_2 \dots x_n).$$

Hence the joint p.d.f of $x_1 x_2 \dots x_n$ is

$$f_\theta(x_1 x_2 \dots x_n) = \varphi_\theta[t_1(x_1 x_2 \dots x_n), \dots t_n(x_1 x_2 \dots x_n)] \times \left|J\left(\frac{t_1 t_2 \dots t_n}{x_1 x_2 \dots x_n}\right)\right|$$

$\Rightarrow f_\theta(x_1, x_2 \ldots x_n) = g_\theta[t_1(x_1, x_2 \ldots x_n)] \cdot h(x_1, x_2 \ldots x_n)$, ..

$= g_\theta(t_1)\, h(x_1, x_2 \ldots x_n)$, (say)

where $g_\theta(t_1) = g_\theta[t_1(x_1, x_2 \ldots x_n)]$

and $h(x_1, x_2 \ldots x_n) = h_1(t_1, t_2 \ldots t_n / t_1) \times \left| J\left(\frac{t_1 t_2 \ldots t_n}{x_1 x_2 \ldots x_n}\right) \right|$

Thus the necessary condition of factorization theorem is established.

Sufficient condition

Suppose the factorization criterion holds. Then we have

$f_\theta(x_1, x_2 \ldots x_n) = g_\theta(t_1)\, h(x_1, x_2 \ldots x_n)$.

Now let us consider the one to one transformation

$$(x_1, x_2 \ldots x_n) \longrightarrow (t_1, t_2 \ldots t_n).$$

Hence the joint p.d.f of $T_1 T_2 \ldots T_n$ is

$$\Psi_\theta(t_1 t_2 \ldots t_n) = f_\theta[x_1(t_1 t_2 \ldots t_n), x_2(t_1 t_2 \ldots t_n) \ldots x_n(t_1 t_2 \ldots t_n)] \times \left| J\left(\frac{x_1 x_2 \ldots x_n}{t_1 t_2 \ldots t_n}\right) \right|$$

$$= g_\theta(t_1) \times h[x_1(t_1 t_2 \ldots t_n) \ldots, x_n(t_1 t_2 \ldots t_n)] \times \left| J\left(\frac{x_1 x_2 \ldots x_n}{t_1 t_2 \ldots t_n}\right) \right|$$

$$= g_\theta(t_1) \times C(t_1 t_2 \ldots t_n)$$

where $C(t_1 t_2 \ldots t_n) = h[x_1(t_1 t_2 \ldots t_n) \ldots, x_n(t_1 t_2 \ldots t_n)] \times \left| J\left(\frac{x_1 x_2 \ldots x_n}{t_1 t_2 \ldots t_n}\right) \right|$

Now the marginal p.d.f of T_1 is obtain from this joint p.d.f of $t_1 t_2 \ldots t_n$ by integrating out $t_2 t_3 \ldots t_n$ and hence is of the form.

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$$g_\theta(t_1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} c(t_1, t_2 \cdots t_n)\prod_{i=2}^{n} dt_i$$

$$= g_\theta(t_1)\, K(t_1).$$

Hence the conditional p.d.f of $T_2 T_3 \cdots T_n$ given $T_1 = t_1$

is $= c(t_1 t_2 \cdots t_n)/K(t_1)$.

which is independent of θ.

Hence the sufficient condition is established.