Multiple correlation

The product moment correlation between x_1 and the joint influence of $x_2, x_3, \dots, x_p$ on x_1 is called multiple correlation coefficient and is denoted by,

$$r_{1.23\cdots p} = \frac{cov(x_1, X_{1.23\cdots p})}{\sqrt{Var(X_1)\,Var(X_{1.23\cdots p})}}$$

Let $Var(X_i) = \sigma_i^2$ then we have

$$X_{1.23\cdots p,\alpha} = \bar{x}_{1,\alpha} - \frac{\sigma_1}{\sigma_2}\frac{R_{12}}{R_{11}}(x_{2,\alpha} - \bar{x}_2) - \cdots$$

$$\cdots - \frac{\sigma_1}{\sigma_p}\frac{R_{1p}}{R_{11}}(x_{p,\alpha} - \bar{x}_p).$$

Summing over the n values of the variates and dividing by n we get

$$\overline{X_{1.23\cdots p}} = \bar{x}_1$$

Now since $x_{1\alpha} = X_{1.23\cdots p\alpha} + e_{1.23\cdots p\alpha}$

then $cov(x_1, X_{1.23\cdots p}) = Var(X_{1.23\cdots p})$

[Since the error $x_{1.23\cdots p,\alpha}$ is uncorrelated with $X_{1.23\cdots}$

$$\therefore r_{1.23\cdots p} = \frac{\sqrt{Var(X_{1.23\cdots p})}}{\sqrt{Var(X_1)}}$$

$$= \frac{\sqrt{Var(X_{1.23\cdots p})}}{\sigma_1}$$

Now. $cov(x_1, X_{1.23\cdots p})$

$$= \frac{1}{n}\sum_{\alpha}(x_{1\alpha} - \bar{x}_1)(X_{1.23\cdots p,\alpha} - \bar{x}_1)$$

[sum is taken over all n possible values of the variables.

$$= -\frac{1}{n}\sum(x_{1a}-\bar{x}_1)\left\{\frac{\sigma_1}{\sigma_2}\frac{R_{12}}{R_{11}}(x_{2a}-\bar{x}_2)+\frac{\sigma_1}{\sigma_3}\frac{R_{13}}{R_{11}}(x_{3,a}-\bar{x}_3)+\cdots+\frac{\sigma_1}{\sigma_p}\frac{R_{1p}}{R_{11}}(x_{p,a}-\bar{x}_p)\right\}$$

$$= -\left[\sigma_1^2\frac{R_{12}}{R_{11}}r_{12}+\sigma_1^2\frac{R_{13}}{R_{11}}r_{13}+\cdots+\sigma_1^2\frac{R_{1p}}{R_{11}}r_{1p}\right]$$

$$= -\frac{\sigma_1^2}{R_{11}}\left[r_{12}R_{12}+r_{13}R_{13}+\cdots+r_{1p}R_{1p}\right]$$

~~Now since $|R|$~~

Now if we consider the diterminent

$$R = \begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ r_{21} & r_{22} & \cdots & r_{2p} \\ \vdots & \vdots & & \vdots \\ r_{p1} & r_{p2} & \cdots & r_{pp} \end{vmatrix}_{p\times p}$$

$$\Rightarrow R = r_{11}R_{11}+r_{12}R_{12}+r_{13}R_{13}+\cdots+r_{1p}R_{1p}$$

$$\Rightarrow r_{12}R_{12}+r_{13}R_{13}+\cdots+r_{1p}R_{1p} = R - r_{11}R_{11}$$

$$= R - R_{11} \quad [\text{Since } r$$

Hence $\operatorname{cov}(X_1, X_{1\cdot 23\cdots p}) = -\frac{\sigma_1^2}{R_{11}}(R - R_{11})$

$$= \sigma_1^2\left(1-\frac{R}{R_{11}}\right)$$

Hence $r_{1\cdot 23\cdots p} = \sqrt{1-\frac{R}{R_{11}}}$

We have $x_1 = X_{1\cdot 23\cdots p} + e_{1\cdot 23\cdots p}$

$\therefore \operatorname{Var}(e_{1\cdot 23\cdots p}) = \operatorname{Var}(x_1) - \operatorname{Var}(X_{1\cdot 23\cdots p})$

or error variance $= \sigma_1^2 - \sigma_1^2\left(1-\frac{R}{R_{11}}\right)$

or, $S^2_{1\cdot 23\cdots p} = \sigma_1^2\frac{R}{R_{11}}$

$\therefore r_{1\cdot 23\cdots p} = \sqrt{1-\frac{S^2_{1\cdot 23\cdots p}}{\sigma_1^2}}$