

~~Ques~~ A dielectric sphere of relative permittivity K is placed in vacuum, where the electric field was initially uniform and given by $E_0 \hat{z}$. The polarization of the sphere is expressed by

$$\vec{P} = 3\epsilon_0 \left(\frac{K-1}{K+2} \right) E_0 \hat{z}.$$

Determine the volume and surface densities of polarization charge. Hence calculate the total polarization charge.

→ Volume & Density of Polarization charge (ρ_p)

The volume bound charge density is defined as : $\rho_p = -\vec{\nabla} \cdot \vec{P}$.

The polarization vector $\vec{P}$ inside the sphere is completely uniform. It is constant in magnitude and points strictly along the z -axis.

The divergence of any constant vector field is zero.

$$\rho_p = - \left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right) = 0.$$

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The surface bound charge density is given by :

$$\sigma_p = \vec{P} \cdot \hat{n}$$

where $\hat{n}$ is the outward unit normal vector.

For a spherical surface, the outward normal ⁽²⁾ is the radial unit vector $\hat{r}$.

In spherical coordinates, the unit vector relationship is $\hat{z} \cdot \hat{r} = \cos \theta$.

Substituting $\vec{P}$ into the equation we get

$$\sigma_p = \left[3\epsilon_0 \left(\frac{K-1}{K+2} \right) E_0 \hat{z} \right] \cdot \hat{r}$$

$$\boxed{\sigma_p = \frac{3\epsilon_0 (K-1) E_0 \cos \theta}{K+2}}$$

Total Polarization charge (q_p)

The total polarization charge is the integral of the densities over the sphere:

$$q_p = \int_{\text{Volume}} \rho_p dV + \int_S \sigma_p dA$$

Since $\rho_p = 0$, we only integrate the surface charge density over the sphere using

$$dA = R^2 \sin \theta d\theta d\phi$$

$$q_p = \int_0^{2\pi} \int_0^\pi \left[\frac{3\epsilon_0 (K-1) E_0 \cos \theta}{K+2} \right] R^2 \sin \theta d\theta d\phi$$

Pulling the constants out of the integration:

$$q_p = \frac{3\epsilon_0 (K-1) E_0 R^2}{K+2} \left(\int_0^{2\pi} d\phi \right) \left(\int_0^\pi \sin \theta \cos \theta d\theta \right)$$

Now $\int_0^{2\pi} d\phi = 2\pi$ and $\int_0^\pi \sin\theta \cos\theta d\theta = \left[\frac{\sin^2\theta}{2} \right]_0^\pi = 0$ (3)

Since the θ integral is zero, the total charge vanishes i.e. $q_p = 0$.

$\therefore$ Total polarization charge $q_p = 0$.

Consider A sphere of radius a made of an inhomogeneous dielectric whose dielectric constant is $K = \frac{1}{(1 + Ar)}$, where A is a constant and r is the radial distance. A charge q is placed at the centre of the sphere. Find D , E and P at any internal point. Also find the volume and the surface densities of polarization charge.

→ Due to the spherical symmetry of the central point charge q , we apply Gauss's law for the electric displacement vector $\vec{D}$ over a concentric spherical Gaussian surface of radius

$r (r < a)$: $\oint \vec{D} \cdot d\vec{A} = q$

Since $\vec{D}$ is radially symmetric and uniform over the spherical shell:

$$D \cdot (4\pi r^2) = q$$

$$\Rightarrow D = \frac{q}{4\pi r^2}$$

In vector form pointing radially outward

$$\vec{D} = \frac{q}{4\pi r^2} \hat{r}$$

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The relationship between the displacement vector $\vec{D}$ and the electric field $\vec{E}$ in any dielectric medium is given by:

$$\vec{D} = \epsilon \vec{E} = K \epsilon_0 \vec{E}$$

Rearranging to solve for $\vec{E}$ and substituting the given position-dependent dielectric constant $K = \frac{1}{1+Ar}$

$$\vec{E} = \frac{\vec{D}}{K \epsilon_0} = \frac{\frac{q}{4\pi r^2} \hat{r}}{\left(\frac{1}{1+Ar}\right) \epsilon_0}$$

$$\vec{E} = \frac{q(1+Ar)}{4\pi \epsilon_0 r^2} \hat{r}$$

The polarization vector $\vec{P}$ represents the induced dipole moment per unit volume and is related to $\vec{D}$ and $\vec{E}$ by

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E}$$

Substituting the expressions we found for $\vec{D}$ and $\vec{E}$

$$\vec{P} = \frac{q}{4\pi r^2} \hat{r} - \epsilon_0 \left[\frac{q(1+Ar)}{4\pi \epsilon_0 r^2} \hat{r} \right]$$

$$\Rightarrow \vec{P} = \frac{q}{4\pi r^2} [1 - (1+Ar)] \hat{r} = \frac{q}{4\pi r^2} (-Ar) \hat{r}$$

$$\Rightarrow \vec{P} = -\frac{qA}{4\pi r} \hat{r}$$

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volume density of polarization charge (ρ_p)

The bound volume charge density is equal to the negative divergence of the polarization vector field

$$\rho_p = -\vec{\nabla} \cdot \vec{P}$$

In spherical coordinates, because $\vec{P}$ only contains a radial component (P_r), the divergence formula simplifies to

$$\vec{\nabla} \cdot \vec{P} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 P_r)$$

Substituting our radial polarization component

$$P_r = -\frac{QA}{4\pi r}$$

$$\vec{\nabla} \cdot \vec{P} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \left(-\frac{QA}{4\pi r} \right) \right]$$

$$\rightarrow \vec{\nabla} \cdot \vec{P} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(-\frac{QA}{4\pi} r \right)$$

Taking the derivative with respect to r :

$$\vec{\nabla} \cdot \vec{P} = \frac{1}{r^2} \left(-\frac{QA}{4\pi} \right) = -\frac{QA}{4\pi r^2}$$

Therefore, the volume charge density is

$$\rho_p = - \left(-\frac{QA}{4\pi r^2} \right)$$

$$\rightarrow \rho_p = \frac{QA}{4\pi r^2}$$

Surface density of Polarization charge (σ_p)

The bound surface charge density accumulated at the boundary interface of the sphere ($r=a$) is given by $\sigma_p = \vec{P} \cdot \hat{n}$

where $\hat{n} = \hat{r}$ is the outward unit normal vector (6) pointing away from the dielectric material at the surface.

Evaluating $\vec{P}$ at the radius $r=a$:

$$\sigma_p = \vec{P}(a) \cdot \hat{r} = \left(-\frac{QA}{4\pi a} \hat{r} \right) \cdot \hat{r}$$

$$\Rightarrow \sigma_p = -\frac{QA}{4\pi a}$$